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NEW HYPERGEOMETRIC-LIKE SERIES FOR $1/\pi^2$ ARISING FROM RAMANUJAN'S THEORY OF ELLIPTIC FUNCTIONS TO ALTERNATIVE BASE 3

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Dedicated to Professor Bruce C. Berndt on the occasion of his 70th birthday

ABSTRACT. By using certain representations for Eisenstein series, we find new hypergeometric-like series for $1/\pi^2$ arising from Ramanujan's theory of elliptic functions to alternative base 3.

1. INTRODUCTION

Let $(a)_0 = 1$ and, for a positive integer n,

$$(a)_n := a(a+1)(a+2)...(a+n-1)$$

and

$$_pF_{p-1}(a_1,...,a_p;b_1,...,b_{p-1};x) := \sum_{n=0}^{\infty} \frac{(a_1)_n...(a_p)_n}{(b_1)_n...(b_{p-1})_n} \frac{x^n}{n!}, \ |x| < 1.$$

In his monumental paper "Modular equations and approximations to π" [14], [15, pp. 23–39], S. Ramanujan offered 17 beautiful series representations for $1/\pi$. Three of his series belong to the classical theory of elliptic functions, while the remaining fourteen series depend on Ramanujan's alternative theories of elliptic functions in which the classical nome "q" is replaced by

$$q_r := q_r(x) := \exp\left(-\pi \csc(\pi/r) \frac{{}_2F_1(\frac{1}{r}, \frac{r-1}{r}; 1; 1-x)}{{}_2F_1(\frac{1}{r}, \frac{1-r}{r}; 1; x)}\right), \tag{1.1}$$

where $r = 3$, 4, or 6. In particular, two of these series

$$\frac{27}{4\pi} = \sum_{k=0}^{\infty} (15k+2) \frac{\left(\frac{1}{2}\right)_k \left(\frac{1}{3}\right)_k \left(\frac{2}{3}\right)_k}{(k!)^3} \left(\frac{2}{27}\right)^k \tag{1.2}$$

and

$$\frac{15\sqrt{3}}{2\pi} = \sum_{k=0}^{\infty} (33k+4) \frac{\left(\frac{1}{2}\right)_k \left(\frac{1}{3}\right)_k \left(\frac{2}{3}\right)_k}{(k!)^3} \left(\frac{4}{125}\right)^k, \tag{1.3}$$

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